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Real Annihilating Measures for $R(K)$

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INTRODUCTION

Let K be a compact subset of the complex plane $\mathbf{C}$, and let $R(K)$ denote the algebra of all continuous complex-valued functions on K which can be approximated uniformly on K by rational functions whose poles all lie outside K. Recently much work has centred on the properties of $R(K)$ as a function algebra, and the study of the annihilating measures has played a central part. See [1]–[7]. The present paper studies these measures using potential theoretic methods based on those of [6], [7], [8].

The main questions we consider are

(i) When does $R(K) = A(K)$, the algebra of continuous functions on K which are analytic on the interior of K?

(ii) Which real-valued functions on K are uniform limits of real parts of functions in $R(K)$?

(iii) When is $A(K)$ a maximal subalgebra of $C(\partial K)$?

In Section 1 the potential theoretic machinery is set up.

In Section 2 we obtain conditions for the uniqueness of Arens-Singer measures, which lead to maximality theorems improving slightly those of [7]. A converse is obtained in Section 3. Sections 2 and 3 rely heavily on the ideas and results of [7].

Section 4 collects some elementary facts about certain annihilating measures and in Section 5 the case where K has connected boundary is considered; conditions are given under which $R(K)$ is dirichlet (i.e. all real-valued functions on ∂K are uniform limits of real parts of functions in $R(K)$).

These results are extended to the general case in Section 6.

Some illustrative examples are given in Section 7.

Theorem 2.2 has been proved independently by T. W. Gamelin.

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Notations and Definitions

Throughout K shall denote a compact plane set, and we put $X = \partial K$ (∂ denotes topological boundary relative to $\mathbf{C}$), $\Omega = \mathbf{C} - K$, $K^0 =$ the interior of K, and denote by Ω_0, Ω_1,... the components of Ω, Ω_0 being the unbounded one. $I = X - \bigcup_i \partial\Omega_i$ is the "inner boundary" of K. We also adopt the following notations as used in [7]: $H(K)$, $\bar{H}(K)$, $D(K)$, P, R. (But note that our $R(K)$ is there called $\bar{0}(K)$).

By "measure" we mean "complex Borel measure" and all measures considered will be suported on compact plane sets. By an "annihilating measure" for $R(K)$ we mean a measure μ on X satisfying

$$\int f \, d\mu = 0, \qquad \forall f \in R(K)$$

If $y \in K$ we define a representing measure for y w.r.t. $R(K)$ to be a positive measure λ on X satisfying

$$f(y) = \int f \, d\lambda, \qquad \forall f \in R(K).$$

We denote the set of annihilating measures by $R(K)^\perp$, and the set of real measures in $R(K)^\perp$ by $\operatorname{Re} R(K)^\perp$. The terms Arens-Singer measure and Jensen measure will be used as defined in [7]. Analogous notations will be used for the other function spaces defined above, but if no space is specified, $R(K)$ will be understood.

We shall occasionally consider $R(L)$ where L is a compact subset of the complex sphere; the definition is obvious.

1. *Potential theoretic preliminaries.*

If $y \in K^0$ we denote by λ_y, the harmonic measure of y w.r.t. K^0 ([10], 8.3). λ_y is a positive measure on X of total mass 1, and is a representing measure for y w.r.t. each of $R(K)$, $A(K)$, $\bar{H}(K)$, $D(K)$; it is also a Jensen measure w.r.t. $A(K)$ (and hence also $R(K)$) by [7], 7.3.

Our main tool will be the logarithmic potential P_μ associated with a real measure μ as follows:

$$P_\mu(z) = \int \ln | z - \zeta |^{-1} \, d\mu(\zeta).$$

If μ is positive (with compact support), then P_μ is defined for all $z \in \mathbf{C}$ as an extended real-valued function, and is superharmonic on $\mathbf{C}$ ([10], 4.1). If μ is not positive we use this notation for such z that $P_{|\mu|}(z) < \infty$ so that the integral defining $P_\mu(z)$ converges.

The following elementary and well-known lemma is the basic link with the potential theory.

1.1 LEMMA. *Let μ be a real measure on X. Then $\mu \in R(K)^{\perp}$ if and only if P_μ is constant on Ω_i for each i.*

Proof. Suppose first $\mu \in R(K)^{\perp}$. Fix i and let $z_0 \in \Omega_i$. Put

$$E = \{z \in \Omega_i \mid P_\mu(z) = P_\mu(z_0)\}.$$

Since P_μ is continuous on Ω, E is closed in Ω_i. Since Ω_i is connected we need only show that E is open to deduce that P_μ is constant on Ω_i. Hence let $z_1 \in E$ and let

$$U = \{z \in \Omega_i \mid | z - z_1 | < d(z_1, K)\}.$$

Then for $z \in U$, $\zeta \in K$ we have

$$\begin{aligned} \ln | z - \zeta |^{-1} - \ln | z_1 - \zeta |^{-1} &= - \ln | 1 - (z_1 - \zeta)^{-1} (z_1 - z)| \\ &= \operatorname{Re} \sum_{r=1}^{\infty} r^{-1}\{(z_1 - z)(z_1 - \zeta)^{-1}\}^r \end{aligned}$$

where for fixed z the series converges uniformly for $\zeta \in K$ so that the sum belongs to $R(K)$, as a function of ζ. Thus

$$\begin{aligned} &\int \ln | z - \zeta |^{-1} d\mu(\zeta) - \int \ln | z_1 - \zeta |^{-1} d\mu(\zeta) \\ &= \operatorname{Re} \int \sum_{r=1}^{\infty} r^{-1}\{(z_1 - z)(z_1 - \zeta)^{-1}\}^r d\mu(\zeta) = 0 \end{aligned}$$

Hence

$$P_\mu(z) = P_\mu(z_1) = P_\mu(z_0)$$

and so $z \in U$. Thus $U \subseteq E$ and E is open as required.

Conversely, suppose P_μ is constant on Ω_i for each i. Let $z_0 \in \Omega$, say $z_0 \in \Omega_i$. If $| z - z_0 | < d(z_0, K)$ we have

$$\ln | z - \zeta |^{-1} - \ln | z_0 - \zeta |^{-1} = \operatorname{Re} \sum_{r=1}^{\infty} r^{-1}\{(z_0 - z)(z_0 - \zeta)^{-1}\}^r$$

and so for all such z,

$$\operatorname{Re} \int \sum_{r=1}^{\infty} r^{-1}\{(z_0 - z)(z_0 - \zeta)^{-1}\}^r d\mu(\zeta) = 0.$$

The integral is an analytic function of z whose real part vanishes near z_0, so the coefficient of each positive power of z vanishes, i.e.

$$\int (z_0 - \zeta)^{-r}\, d\mu(\zeta) = 0, \qquad r = 1, 2, \ldots .$$

It remains to show that μ annihilates polynomials; for $|z|$ large we have

$$(\zeta - z)^{-1} = -\sum_{n=0}^{\infty} \zeta^n z^{-(n+1)}, \qquad \zeta \in K$$

so that for all large z

$$\sum_{n=0}^{\infty} z^{-(n+1)} \int \zeta^n\, d\mu(\zeta) = 0.$$

By the uniqueness of the Laurent expansion each coefficient must be zero, which completes the proof.

1.2 COROLLARY. *Let μ be a positive measure on X and let $y \in K$. Then μ is a representing measure for y if and only if $g(z)$ is constant on Ω_i for each i, where*

$$g(z) = P_\mu(z) - \ln |z - y|^{-1}.$$

Proof. Let $\nu = \lambda_y$ or δ_y (point mass at y) according as $y \in K^0$ or $y \in X$. Then for $z \in \Omega$ we have

$$P_\nu(z) = \ln |z - y|^{-1}$$

and μ is a representing measure for y if and only if $\mu - \nu \in R(K)^\perp$ so the result follows from Lemma 1.1.

Notes. 1. If $\mu \in R(K)^\perp$ then since $P_\mu \to 0$ as $z \to \infty$, we have $P_\mu = 0$ on Ω_0. Similarly, $P_\mu(z) = \ln |z - y|^{-1}$ for $z \in \Omega_0$ if μ is a representing measure for y.

2. The above arguments apply to measures on K, not necessarily supported on X, but we shall not consider such measures.

3. For the special case of harmonic measure, we have as noted above $P_{\lambda_y}(z) = \ln |z - y|^{-1}$, $y \in K^0$, $z \in \Omega$. It is less obvious that $P_{\lambda_y}(z) = \ln |z - y|^{-1}$, $z \in R$ (the set of regular points w.r.t. K^0 on ∂K); in fact, this property characterises the set of regular points. See [10], 9.2 and 9.3.

We next obtain analogous conditions for Arens-Singer measures.

1.3. LEMMA. *Let μ be a real measure on X. Then the following are equivalent:*

(i) $P_\mu(z) = 0$, $\forall z \in \Omega$

(ii) $\mu \in \overline{H}(K)^\perp$

(iii) *Whenever f is an invertible element of $R(K)$,*

$$\int \ln |f| \, d\mu = 0.$$

Proof. (ii) $\Rightarrow$ (iii). Let f be an invertible element of $R(K)$ (i.e. $f \in R(K)$ and $f^{-1} \in R(K)$). Then we can find a sequence $\{g_n\}$ of functions analytic and non-vanishing in a neighborhood of K, with $g_n \to f$ uniformly on K.

Then $\ln |g_n| \in H(K)$ so that $\int \ln |g_n| \, d\mu = 0$ by (ii). Further, $\ln |g_n| \to \ln |f|$ uniformly on K, and hence $\int \ln |f| \, d\mu = 0$.

(iii) $\Rightarrow$ (i). Since $|z - \zeta|^{-1}$ is invertible in $R(K)$ (for z fixed),

$$P_\mu(\zeta) = \int \ln |z - \zeta|^{-1} \, d\mu(\zeta) = 0.$$

(i) $\Rightarrow$ (ii). Let $f \in H(K)$; we may suppose f extended to be twice continuously differentiable on $\mathbf{C}$, with compact support; then

$$\begin{aligned}
\int f(\zeta) \, d\mu(\zeta) &= \frac{1}{2\pi} \int \left(\int \nabla^2 f(z) \ln |z - \zeta|^{-1} \, dm(z) \right) d\mu(\zeta) \\
&= \frac{1}{2\pi} \int \left(\int \ln |z - \zeta|^{-1} \, d\mu(\zeta) \right) \nabla^2 f(z) \, dm(z) \\
&= \frac{1}{2\pi} \int P_\mu(z) \nabla^2 f(z) \, dm(z) \\
&= 0
\end{aligned}$$

where m denotes plane Lebesque measure and the integrals converge absolutely since $\ln |z|^{-1}$ is integrable (dm).

1.4. COROLLARY. *Let μ be a positive measure on X, and let $y \in K$. Then μ is an Arens-Singer measure for y if and only if, for $z \in \Omega$,*

$$P_\mu(z) = \ln |z - y|^{-1}.$$

Proof. Just as in Corollary 1.2.

By Lemma 2 of [8] a real measure is determined by its logarithmic potential. Thus questions concerning representing measures can be transformed to questions concerning superharmonic functions. We end the section with a lemma which is useful in studying logarithmic potentials; the proof is essentially due to Carleson ([8], Lemma 1).

1.5. LEMMA. *Let U be an open subset of $\mathbf{C}$, and Y and E subsets of U such that E has zero one-dimensional Hausdorff outer measure and $Y \cup E$ is connected. Let μ be a real measure on a compact subset of U so that $P_{|\mu|}$ is finite on Y. Let f be a real continuous function on U and define*

$$g(z) = P_\mu(z) - f(z)$$

wherever it is defined. Further, suppose $g(z)$ a constant λ for $z \in Y$. Suppose $z_0 \in \bar{Y} \cap U$ and $P_{|\mu|}(z_0) < \infty$. Then $g(z_0) = \lambda$.

Proof. The result is trivial if $Y \cup E = \{z_0\}$ so we assume $\exists z_1 \in Y \cup E$, $z_1 \neq z_0$. Suppose $g(z_0) \neq \lambda$ and choose $\epsilon > 0$ so that $|\lambda - g(z_0)| > 4\epsilon$. Since $P_{|\mu|}(z_0) < \infty$ we have $|\mu|(\{z_0\}) = 0$; hence we can choose $\delta > 0$ so that $\delta < \frac{1}{2}$ and

$$\int_{|\zeta - z_0| < \delta} \ln |z_0 - \zeta|^{-1} \, d|\mu|(\zeta) < \epsilon.$$

For any ρ with $0 < \rho < \delta$ choose $\eta(\rho)$ so that

$$0 < \eta(\rho) < \delta, \qquad |z_1 - z_0| > \eta(\rho)$$

and

$$|z - z_0| < \eta(\rho) \Rightarrow z \in U, \qquad |f(z) - f(z_0)| < \epsilon$$

and

$$\int_{|\zeta - z_0| \geqslant \rho} |\ln |\zeta - z_0|^{-1} - \ln |\zeta - z|^{-1}| \, d|\mu|(\zeta) < \epsilon.$$

Given ρ and $\eta(\rho)$ as above, let

$$E_\rho = \{r : 0 < r < \eta(\rho) \text{ and } \exists z \in E \text{ with } |z - z_0| = r\}.$$

The hypothesis on E implies easily that E_ρ has (linear) measure zero. Now let $0 < r < \eta(\rho)$ and $r \notin E_\rho$. Then since $Y \cup E$ is connected, $\exists z \in Y$ with $|z - z_0| = r$.

Then

$$4\epsilon < |g(z) - g(z_0)|$$

$$\leqslant |f(z) - f(z_0)| + \int_{|\zeta - z_0| \geqslant \rho} |\ln|\zeta - z_0|^{-1} - \ln|\zeta - z|^{-1}|\, d|\mu|(\zeta)$$

$$+ \int_{|\zeta - z_0| < \rho} \ln|\zeta - z_0|^{-1}\, d|\mu|(\zeta) + \int_{|\zeta - z_0| < \rho} \ln|\zeta - z|^{-1}\, d|\mu|(\zeta)$$

$$< 3\epsilon + \int_{|\zeta - z_0| < \rho} \ln|\zeta - z|^{-1}\, d|\mu|(\zeta)$$

so

$$\int_{|\zeta - z_0| < \rho} \ln|\zeta - z|^{-1}\, d|\mu|(\zeta) > \epsilon$$

and thus

$$\int_{|\zeta - z_0| < \rho} \ln|r - |\zeta - z_0||^{-1}\, d|\mu|(\zeta) > \epsilon.$$

This is true for all r with $0 < r < \eta(\rho)$ except for a set of measure zero. Hence

$$\eta(\rho)^{-1} \int_0^{\eta(\rho)} \left(\int_{|\zeta - z_0| < \rho} \ln|r - |\zeta - z_0||^{-1}\, d|\mu|(\zeta) \right) dr > \epsilon.$$

But

$$\eta(\rho)^{-1} \int_0^{\eta(\rho)} \ln|r - |\zeta - z_0||^{-1}\, dr$$

$$= T^{-1} \int_0^T \ln|u - 1|^{-1}\, du + \ln|\zeta - z_0|^{-1} \leqslant A + \ln|\zeta - z_0|$$

where A is an absolute constant.

So

$$\int_{|\zeta - z_0| < \rho} (A + \ln|\zeta - z_0|^{-1})\, d|\mu|(\zeta) > \epsilon$$

for all ρ with $0 < \rho < \delta$. This is a contradiction, so $g(z_0) = \lambda$ as required.

Note. This lemma could be proved more quickly assuming more background in potential theory. We have preferred not to do this since it is used in Section 5 which otherwise requires only a bare minimum of potential theory (basic properties of harmonic measures).

2. *Arens-Singer measures.*

2.1. Lemma. *Let $z \in P$ and let f be a superharmonic function in a neighbourhood of z. Then*

$$f(z) = \liminf_{\zeta \in \Omega, \zeta \to z} f(\zeta).$$

Proof. If the conclusion is false, we can find a superharmonic function f in a neighbourhood U of z with $f(z) = -1$ and $f(\zeta) > 1$ for $\zeta \in U \cap \Omega$. By Theorem 6.6 of [7], $\exists g \in \bar{H}(K)$ with $g(z) = 0$, $g(\zeta) < 0$ for $\zeta \in K - \{z\}$. Let U_1 be an open neighbourhood of z with $\bar{U}_1 \subseteq U$; then f is bounded below on $\bar{U}_1$. So $\exists \alpha > 0$ with $\alpha g - f < -1$ on $\partial U_1 \cap K$. Then since $g \in \bar{H}(K)$ we can find $F \in H(K)$ harmonic on an open neighbourhood V of K with $F(z) > -1/\alpha$, $F < 0$ on V and $\alpha F - f < -1$ on $\partial U_1 \cap V$. Let W be an open set with $K \subseteq W \subseteq \bar{W} \subseteq V$; then $\alpha F - f \leqslant -1$ on $\partial(U_1 \cap W)$ and hence on $U_1 \cap W$, which contradicts

$$\alpha F(z) - f(z) > -1 + 1 = 0.$$

2.2. Theorem. *Suppose that $\lambda_y(X - P) = 0$ for all $y \in K^0$ and that $\bar{H}(X) = C_R(X)$. Then $\bar{H}(K) = D(K)$.*

Proof. By [7], Theorem 6.9, it suffices to prove that $P = R$. So let $z_0 \in R$ and let μ be any representing measure for z_0 w.r.t. $\bar{H}(K)$ on X. By Lemma 5.2 of [7], $\mu(X - R) = 0$. Clearly $P_\mu(z) = \ln |z - z_0|^{-1}$ for $z \in \Omega$. By Lemma 2.1 this equation holds for $z \in P$. So for $y \in K^0$,

$$P_\mu(z) = \ln |z - z_0|^{-1} \quad \text{a.e. } (\lambda_y).$$

Also,

$$\ln |y - \zeta|^{-1} = \int \ln |z - \zeta|^{-1} \, d\lambda_y(z) \qquad \text{for} \qquad \zeta \in R,$$

and hence a.e. (μ). Thus

$$\begin{aligned} P_\mu(y) &= \int \ln |y - \zeta|^{-1} \, d\mu(\zeta) \\ &= \iint \ln |z - \zeta|^{-1} \, d\lambda_y(z) \, d\mu(\zeta) \\ &= \int \ln |z - z_0|^{-1} \, d\lambda_y(z) \\ &= \ln |y - z_0|^{-1} \end{aligned}$$

since $z_0 \in R$. Hence by Lemma 1.3, $\delta_{z_0} - \mu \in \bar{H}(X)^{\perp}$ so since $\bar{H}(X) = C_R(X)$, $\mu = \delta_{z_0}$. Thus $z_0 \in P$ and so $R = P$ as required.

2.3. THEOREM. *Let $y_0 \in K^0$. Then y_0 has unique Arens-Singer measure if and only if $\lambda_{y_0}(X - P) = 0$.*

Proof. If $\lambda_{y_0}(X - P) > 0$ then Keldysh measure for y_0 is an Arens-Singer measure distinct from λ_{y_0} by [7] 7.3 and 9.2.

Suppose conversely that $\lambda_{y_0}(X - P) = 0$. Let U be the component of K^0 containing y_0. Then $P_\mu(z) = \ln | z - y_0 |^{-1}$ for $z \in \Omega$ and hence for $z \in P$ by Lemma 2.1. For $y \in U$, $\lambda_y(X - P) = 0$, and $\mu(X - R) = 0$ by [7] 5.2. Hence

$$\begin{aligned} P_\mu(y) &= \int \ln | y - \zeta |^{-1} \, d\mu(\zeta) \\ &= \int\int \ln | z - \zeta |^{-1} \, d\lambda_y(z) \, d\mu(\zeta) \\ &= \int \ln | z - y_0 |^{-1} \, d\lambda_y(z) = P_{\lambda_{y_0}}(y) \end{aligned}$$

(since μ can be replaced by λ_{y_0}). Let $g(z) = P_\mu(z) - \ln | z - y_0 |^{-1}$. Then $g = 0$ in Ω and in U, $g(z) = - G(y_0, z)$, the Green's function of U. (See [10], Ch. 9). Let $Y = K - U$ and let R_1 be the set of regular points for U on ∂U. Since g is superharmonic outside y_0 it is lower semi-continuous and so $g \leqslant 0$ on X. Let $w \in X - (\partial U - R_1)$; we shall prove $g(w) = 0$. So assume $g(w) < 0$ and let φ be a positive superharmonic function on a neighbourhood N of K, with $\varphi = + \infty$ on $\partial U - R_1$ and $h(w) < 0$, where $h = g + \varphi$. Put $\delta = - \frac{1}{2} h(w) > 0$. (For existence of φ see [10], 5.3).

We claim that for $z \in X$, $\exists$ a neighbourhood V of z with $h > - \delta$ in $V \cap (\mathbf{C} - Y)$, and $y_0 \notin V$. If $z \in X - \partial U$ we need only choose $V \subseteq N - \bar{U}$. If $z \in R_1$ then $G(y_0, \zeta) \to 0$ as $\zeta \to z$, $\zeta \in U$ ([10], 9.3). Thus we can choose $V \subseteq N$ so that $f > - \delta$ in $V \cap U$. Finally, if $z \in \partial U - R_1$ then $h(z) = \infty$, so by lower semicontinuity of h we can find V. We can cover X by the interiors of finitely many such sets $V_1 \cdots V_n$ which we may assume to be compact. Let $W = \bigcup_{i=1}^{n} V_i \cup Y$; then W is compact and $\partial W \subseteq \mathbf{C} - Y$, so $h > - \delta$ on ∂W and hence on W since h is superharmonic. This contradicts $h(w) = - 2\delta$, so $g(w) = 0$.

So $g = 0$ on $X - (\partial U - R_1)$ and thus a.e. (λ_y) for all $y \in K^0$. Just as above we deduce $P_\mu(y) = P_{\lambda_{y_0}}(y)$, $y \in K^0$. Hence $P_\mu(z) = P_{\lambda_{y_0}}(z)$ in $\mathbf{C} - (\partial U - R_1)$ which has capacity zero. By [8] Lemma 2, $\mu = \lambda_0$.

2.4. COROLLARY. *Suppose that* $\lambda_y(X - P) = 0$ *for all* $y \in K^0$, *and that* K^0 *is connected. Then if* B *is a closed subalgebra of* $C(X)$ *containing* $R(K)$ *then either* $B \subseteq A(K)$ *or* $B \supseteq R(X)$.

Proof. This follows from Theorem 2.3 and Theorem 2.2 of [7]. (The hypothesis K^0 dense in K is not used in [7], 2.2.)

2.5. THEOREM. *Suppose that* $\lambda_y(X - P) = 0$, $\forall y \in K^0$, *and that* $R = X$. *Then* $\bar{H}(K) \mid X = \bar{H}(X)$.

Proof. Let $\mu \in \bar{H}(K)^\perp$, so that $P_\mu = 0$ in Ω. Put $\mu = \mu_+ - \mu_-$ where μ_+ and μ_- are positive. If $z \in P$ and $P_{|\mu|}(z) < \infty$, then by Lemma 2.1 applied separately to P_{μ_+} and P_{μ_-} we have $P_\mu(z) = 0$. Since $R = X$,

$$\ln | y - \zeta |^{-1} = \int \ln | z - \zeta |^{-1} \, d\lambda_y(z), \qquad \forall \zeta \in X, \qquad y \in K^0.$$

If $y \in K^0$,

$$\begin{aligned} \int P_{|\mu|}(z) \, d\lambda_y(z) &= \int\int \ln | z - \zeta |^{-1} \, d \mid \mu \mid (\zeta) \, d\lambda_y(z) \\ &= \int \ln | y - \zeta |^{-1} \, d \mid \mu \mid (\zeta) < \infty \end{aligned} \tag{1}$$

Hence $P_{|\mu|}(z) < \infty$ a.e. (λ_y) and so $P_\mu(z) = 0$ a.e. (λ_y), since $\lambda_y(X - P) = 0$. Thus

$$\begin{aligned} P_\mu(y) &= \int \ln | y - \zeta |^{-1} \, d\mu(\zeta) \\ &= \int\int \ln | z - \zeta |^{-1} \, d\lambda_y(z) \, d\mu(\zeta) \\ &= \int P_\mu(z) \, d\lambda_y(z) \\ &= 0, \end{aligned}$$

the integrals converging absolutely by (1). So $P_\mu = 0$ outside X and $\mu \in \bar{H}(X)^\perp$. Thus $\bar{H}(K)^\perp = \bar{H}(X)^\perp$ and so $\bar{H}(K) \mid X = \bar{H}(X)$.

3. *Construction of Jensen Measures.*

In this section we establish a converse to the results of Section 2, to the effect that if $y \in K^0$ and $\lambda_y(X - P) > 0$, y possesses an abundance of Jensen measures. One such measure, namely Keldysh measure, is constructed in [7] and our construction is an extension of that given in [7].

3.1. THEOREM. *Suppose* $y_0 \in K^0$ *and* $\lambda_{y_0}(X - P) > 0$. *Let* F *be a closed subset of* X *with* $\lambda_{y_0}(X - P - F) > 0$. *Then* $\exists$ *a Jensen measure* μ *for* y_0 *with* $\mu \mid (X - P - F) = 0$ *and* $\mu(T) \geqslant \lambda_{y_0}(T)$ *for all Borel sets* $T \subseteq F$.

Proof. Let $\{U_n\}$ be a decreasing sequence of bounded open sets with $K - F = \bigcap_n U_n$, $K = \bigcap_n \bar{U}_n$ and $F \subseteq \partial U_n$ for each n. (Such a sequence is easily constructed.) Let λ_n denote the harmonic measure of y_0 w.r.t. U_n on ∂U_n and write $\lambda = \lambda_{y_0}$. By replacing $\{U_n\}$ by a subsequence if necessary, we may assume λ_n converges weak* to a measure μ; we shall show that μ has the required properties. Clearly $\mu \geqslant 0$ and is supported on X.

(I) μ is a Jensen measure for y_0. Let g be analytic in a neighbourhood of K, and hence of $\bar{U}_n$ for n large enough. Then for $\epsilon > 0$, n large enough,

$$\ln |g(y_0)| \leqslant \int \ln |g(\zeta)| \, d\lambda_n(\zeta)$$

$$\leqslant \int \ln(|g(\zeta)| + \epsilon) \, d\lambda_n(\zeta)$$

so

$$\ln |g(y_0)| \leqslant \int \ln(|g(\zeta)| + \epsilon) \, d\mu(\zeta)$$

and by uniform convergence this holds also for $g \in R(K)$. Thus

$$\ln |g(y_0)| \leqslant \int \ln |g(\zeta)| \, d\mu_n(\zeta),$$

for $g \in R(K)$ so μ is a Jensen measure for y_0.

(II) $\mu \mid F \geqslant \lambda_{y_0} \mid F$. First, $\lambda_n \mid F \geqslant \lambda_{y_0} \mid F$ for if E is a Borel subset of F, then by [10], 8.4, $\lambda_n(E) = \inf \varphi(y_0)$ taken over all superharmonic φ on U_n with $\varphi \geqslant 0$ and

$$\liminf_{\zeta \to z, \zeta \in U_n} \varphi(\zeta) \leqslant 1, \qquad \forall z \in F,$$

whilst $\lambda_{y_0}(E)$ is the similar infimum with U_n replaced by K^0 and if φ belongs to the first class its restriction to K^0 belongs to the second. So $\lambda_n(E) \geqslant \lambda_{y_0}(E)$.

Next, if M is a closed subset of F and f is a non-negative continuous function on $\mathbf{C}$ with $f = 1$ on M, then

$$\begin{aligned}\int f\,d\mu &= \lim_n \int f\,d\lambda_n \\ &\geqslant \limsup_n \lambda_n(M) \\ &\geqslant \lambda_{y_0}(M).\end{aligned}$$

So $\mu(M) \geqslant \lambda_{y_0}(M)$ for all closed $M \subseteq F$. Hence $\mu \mid F \geqslant \lambda_{y_0} \mid F$ as required.

(III) $\mu(X - P - F) = 0$. We prove this by constructing a measure ν on $P \cup F$ then showing that $\mu = \nu$.

For $n = 1, 2,...$ let D_n denote the space of functions in $C_R(\bar{U}_n)$ which are harmonic in U_n. Let B denote the uniform closure in $C_R(X)$ of the subspace $\bigcup_n (D_n \mid X)$; then $B \supseteq \bar{H}(K) \mid X$. Let ∂_B denote the Choquet boundary of B, so $\partial_B \subseteq X$. By Choquet's theorem ([11], Ch. 3) $\exists$ a representing measure ν for y_0 w.r.t. B concentrated on ∂_B. We claim that $\mu = \nu$.

Let $f \in C_R(X)$; we require to prove $\int f\,d\mu = \int f\,d\nu$. Extend f to be bounded and continuous on $\mathbf{C}$. Let R_n denote the set of regular points of ∂U_n w.r.t. U_n. Let E_n be an increasing sequence of compact sets whose union is $X - F$. For $y \in U_n$ let λ_{yn} denote the harmonic measure of y w.r.t. U_n. Since $\partial U_n - R_n$ has logarithmic capacity zero, $\nu(\partial U_n - R_n) = 0$ by [7], 5.2. Hence we can find, for each n, a compact subset Q_n of R_n such that

$$\nu(\partial U_n - Q_n) < \frac{1}{n^2}, \qquad \lambda_{yn}(\partial U_n - Q_n) < \frac{1}{n}$$

for $y \in E_n$. Then choose $f_n \in D_n$ with $f_n = f$ on Q_n and $\|f_n\|_{\bar{U}_n} \leqslant 2\|f\|_{\mathbf{C}}$; that such an f_n can be chosen follows by applying [7] 4.5 with D_n whose Choquet boundary is R_n. That the condition on the norm can be satisfied follows from the proof of that theorem.

Since $f_n = f$ on Q_n and $F \subseteq \partial U_n$ for all n we have $f_n \to f$ on the set

$$F - \bigcup_{n=k}^{\infty} (\partial U_n - Q_n)$$

for all k and this set has ν-measure $< \sum_{n=k}^{\infty} 1/n^2$. So $f_n \to f$ on F a.e. (ν). Next let $y \in \partial_B - F$. Then $y \in U_n$ for all n. So if ρ is a weak* limit of any subsequence of $\{\lambda_{yn}\}$ then ρ is a representing measure for

y w.r.t. B and since $y \in \partial_B$, $\rho = \delta_y$. Hence $\lambda_{yn} \to \delta_y$ weak*. For n large enough $y \in F_n$, hence

$$\int (f - f_n)\, d\lambda_{yn} \to 0 \qquad \text{as} \qquad n \to \infty$$

and

$$\begin{aligned}\lim_n f_n(y) &= \lim_n \int f_n\, d\lambda_{yn} \\ &= \int f\, d\delta_y = f(y).\end{aligned}$$

Thus $f_n \to f$ a.e. (ν) on ∂_B. But ν is concentrated on ∂_B so

$$\begin{aligned}\int f\, d\nu &= \lim_n \int f_n\, d\nu \\ &= \lim_n f_n(y_0).\end{aligned}$$

Also, since $y_0 \in E_n$ for n large enough,

$$\begin{aligned}\int f\, d\mu &= \lim_n \int f\, d\lambda_n \\ &= \lim_n \int f_n\, d\lambda_n \\ &= \lim_n f_n(y_0).\end{aligned}$$

So $\int f\, d\mu = \int f\, d\nu$ and $\mu = \nu$ as required.

To complete the proof we must show that $\nu(X - P - F) = 0$. It suffices to show that $\partial_B \subseteq P \cup F$. So let $z \in X - P - F$ and let M be a compact neighbourhood of z with $M \cap F = \phi$. Since $z \notin P$, the local nature of the characterisation of P in [7] 6.6(iv) shows that z does not belong to the set P associated with $M \cap K$ (i.e. the Choquet boundary of $\bar{H}(M \cap K)$). Thus $\exists$ a positive measure ρ on $M \cap K$ representing z w.r.t. $\bar{H}(M \cap K)$ with $\rho \neq \delta_z$. Then ρ represents z w.r.t. B so $z \notin \partial_B$. Thus $\partial_B \subseteq P \cup F$ and the proof is complete.

3.2. THEOREM. *Let $y \in K^0$. The following are equivalent:*

(i) $\lambda_y(X - P) = 0$.

(ii) λ_y *is the only Arens-Singer measure for y.*

(iii) *The set of Jensen measures for y is compact in the norm topology.*

Proof. (i) $\Rightarrow$ (ii) by Theorem 2.3 and (ii) $\Rightarrow$ (iii) is trivial. To prove (iii) $\Rightarrow$ (i), suppose $\lambda_y(X - P) > 0$, and choose ϵ so that $0 < \epsilon < \lambda_y(X - P)$. It is easy to construct inductively a compact set $F_{\epsilon_1 \cdots \epsilon_n}$ for each sequence $\epsilon_1 \cdots \epsilon_n$, $\epsilon_i = 0$ or 1 with the following properties: F_0, F_1 are disjoint subsets of $X - P$, $\lambda_y(F_0) > \epsilon/2$, $\lambda_y(F_1) > \epsilon/2$, $F_{\epsilon_1 \cdots \epsilon_{n-1} 1}$ and $F_{\epsilon_1 \cdots \epsilon_{n-1} 0}$ are disjoint subsets of $F_{\epsilon_1 \cdots \epsilon_{n-1}}$, and $\lambda_y(F_{\epsilon_1 \cdots \epsilon_n}) > \epsilon/2^n$ for $n > 1$.

For each n put $F_n = \bigcup_{\epsilon_1 \cdots \epsilon_{n-1}} F_{\epsilon_1 \cdots \epsilon_{n-1} 0}$ so that F_n is compact and $\lambda_{y_0}(F_n) > \epsilon/2$. By Theorem 3.1 we can find a Jensen measure λ_n with $\lambda_n \mid F_n \geqslant \lambda_0 \mid F_n$ and $\lambda_n(X - P - F_n) = 0$. Suppose $m < n$: for each sequence $\epsilon_1 \cdots \epsilon_m$ either $\lambda_m(F_{\epsilon_1 \cdots \epsilon_m}) = 0$ or $\lambda_m \geqslant \lambda_{y_0}$ on $F_{\epsilon_1 \cdots \epsilon_m}$. In either case it is clear that

$$\| \lambda_m - \lambda_n \mid F_{\epsilon_1 \cdots \epsilon_m} \| > \epsilon\, 2^{-m-1}.$$

Hence

$$\| \lambda_m - \lambda_n \| > \frac{\epsilon}{2}$$

whenever $m < n$. Thus the set of Jensen measures for y cannot be norm compact.

4. *Certain real annihilating measures.*

In this section we recall briefly an elementary and well-known construction for real annihilating measures. Later we shall examine to what extent these exhaust $\operatorname{Re} R(K)^{\perp}$.

Let Γ be a piecewise smooth Jordan curve in K^0. We define a real measure $\lambda(\Gamma)$ on X as follows: for $f \in C_R(X)$,

$$\int f \, d\lambda(\Gamma) = -\frac{1}{2\pi} \int_\Gamma \frac{\partial \tilde{f}}{\partial n} \, ds$$

where $\tilde{f}$ is the harmonic extension of f to K^0, n is the outward normal and s is arc length. It is elementary that $f \to \int f \, d\lambda(\Gamma)$ indeed defines a bounded linear functional on $C_R(X)$ and that moreover

$$\left| \int f \, d\lambda(\Gamma) \right| \leqslant M \int |f| \, d\lambda_y$$

where y is any point in the component of K^0 containing Γ and M depends only on K, Γ, and the choice of y. $\int f \, d\lambda(\Gamma)$ is the period of the harmonic conjugate of $\tilde{f}$ about Γ, hence is zero if $f \in A(K)$ i.e.

$\lambda(\Gamma) \in A(K)^{\perp}$ (and hence $\in R(K)^{\perp}$). Further $\lambda(\Gamma) \geqslant 0$ on the part of X inside Γ and $\leqslant 0$ on the part outside (see [12] III 4c).

It is convenient to consider certain linear combinations of the $\lambda(\Gamma)$. Let Q be a bounded component of $\bar{\Omega}$ such that $Q \cap (\bar{\Omega} - Q)^{-} = \phi$. Then only finitely many components of $C - Q$ meet $\bar{\Omega}$, say $U_0, U_1 \cdots U_n$ where U_0 is unbounded. For $i = 0, 1 \cdots n$ let Γ_i be a smooth Jordan curve in K^0 separating Q from $U_i \cap \bar{\Omega}$. Then the measure

$$\lambda = \lambda(\Gamma_0) - \sum_{i=1}^{n} \lambda(\Gamma_i)$$

is easily shown to satisfy $P_\lambda = 1$ on $\Omega \cap Q$ and $P_\lambda = 0$ on $\Omega - Q$. λ is positive on $Q \cap X$ and negative on $X - Q$. We denote this measure λ by $\lambda(Q)$.

5. *Annihilating measures-$\bar{\Omega}$ connected.*

In this section we assume throughout that $\bar{\Omega}$ is connected. Roughly speaking we show that if I is small enough then $R(K)$ is Dirichlet (i.e. $\operatorname{Re} R(K)^{\perp} = \{0\}$). We do this by using the characterisation of representing measures in section one, together with arguments based on those of Carleson [8] and McCullough [6], that interior points of K have unique representing measures, and then applying results of function-algebra theory to deduce that $R(K)$ is dirichlet. This section requires no potential theoretic background beyond elementary properties of harmonic measure.

5.1. Lemma (McCullough [6]). *Let Y be a compact connected subset of $\bar{\Omega}$ containing more than one point such that $Y \cap I$ is countable. Let $y \in K^0$ and let μ be a representing measure for y on X. Let*

$$g(z) = P_\mu(z) - \ln | z - y |^{-1}.$$

Then g is a finite constant on Y.

Proof. By 1.2, g is constant, say λ_i, on Ω_i for each i. Let Q_i be the component of the set $\{g = \lambda_i\}$ containing Ω_i. If $z_0 \in \bar{Q}_i$ and $z_0 \neq y$ then $P_\mu(z_0) < \infty$ since P_μ is lower semi-continuous. Hence by Lemma 1.5 (with $U = \mathbf{C} - \{y\}$), $\bar{Q}_i$ is closed in $\mathbf{C} - \{y\}$ and in particular $\bar{\Omega}_i \subseteq Q_i$. Any two of the sets $Q_i \cap Y$, $i = 1, 2, \ldots$, are either identical or disjoint; moreover $Y - \bigcup_i (Q_i \cap Y)$ is countable (being a subset of I). But a compact connected set cannot be a countable disjoint union of closed subsets unless all but one are empty ([13], 42 III 6). Since Y does not reduce to a point, we must have $Y \subseteq Q_i$ for some i i.e. $g = \lambda_i$ on Y.

5.2. LEMMA. *Let* $0 < \epsilon < 1$, $0 < \alpha < \frac{1}{4}$, $0 < x < \frac{1}{4}$. *Then*

$$\int_\beta^\alpha \epsilon \ln |x - r|^{-1} (r \ln r^{-1})^{-1}\, dr < (1 + A\epsilon) \ln x^{-1}$$

where A is an absolute constant and

$$\ln \ln \beta^{-1} = \epsilon^{-1} + \ln \ln \alpha^{-1}.$$

Proof. Let

$$f(r) = \epsilon \ln |x - r|^{-1} (r \ln r^{-1})^{-1}$$

(a) If $2x \leqslant \alpha$, putting

$$\ln |x - r|^{-1} = \ln r^{-1} - \ln |1 - r^{-1}x|$$

gives

$$\begin{aligned}\int_{2x}^\alpha f(r)\, dr &\leqslant \int_{2x}^\alpha \epsilon r^{-1}\, dr + \int_{2x}^\alpha 2\epsilon x r^{-2} (\ln r^{-1})^{-1}\, dr \\ &\leqslant \epsilon \ln(2x)^{-1} + 2\epsilon x \int_{2x}^\alpha r^{-2}\, dr \\ &< 2\epsilon \ln x^{-1}\end{aligned}$$

(b) $$\begin{aligned}\int_{\frac{1}{2}x}^{2x} f(r)\, dr &\leqslant 2\epsilon (x \ln(2x)^{-1})^{-1} \int_{\frac{1}{2}x}^{2x} \ln |x - r|^{-1}\, dr \\ &= 2\epsilon (x \ln(2x)^{-1})^{-1} x \left[\tfrac{3}{2} \ln x^{-1} + \int_{\frac{1}{2}}^{2} \ln |s - 1|^{-1}\, ds\right] \\ &\leqslant A_1 \epsilon \ln x^{-1}.\end{aligned}$$

(c) If $\beta \leqslant x/2$ and $\eta = \min(\frac{1}{2}x, \alpha)$ then

$$\begin{aligned}\int_\beta^\eta f(r)\, dr &\leqslant \int_\beta^\alpha \epsilon \ln x^{-1} (r \ln r^{-1})^{-1}\, dr + \int_\beta^{\frac{1}{2}x} \epsilon \ln |x(x - r)^{-1}| (r \ln r^{-1})^{-1}\, dr \\ &\leqslant \ln x^{-1} + 2\epsilon x^{-1} \int_\beta^{\frac{1}{2}x} (\ln r^{-1})^{-1}\, dr \\ &\leqslant \ln x^{-1} + \epsilon.\end{aligned}$$

Putting (a), (b), (c) together gives the result.

5.3. LEMMA. *Let* $0 < \epsilon < 1$, $\alpha \geqslant \frac{1}{4}$, $\alpha < a$ *and*

$$\ln \ln \beta^{-1} = \ln \ln \alpha^{-1} + \epsilon^{-1}.$$

Let $z_0 \in \mathbf{C}$ *and let* μ *be a non-zero positive measure on a compact subset of* $\{z : |z - z_0| < \alpha\}$. *Then* $\exists r$, $\beta < r < \alpha$, *such that for* $|z - z_0| = r$,

$$P_\mu(z) < (1 + A\epsilon) P_\mu(z_0) + A\epsilon \ln 4a$$

where A is the constant of Lemma 5.2.

Proof. We may suppose $z_0 = 0$; also the transformation $z = 4az'$ reduces to the case $a = \frac{1}{4}$ so we suppose $a = \frac{1}{4}$. Suppose that, $\forall r$ with $\beta < r < \alpha$, $\exists z$ with $|z| = r$ and

$$P_\mu(z) \geqslant (1 + A\epsilon) P_\mu(0).$$

For such r,

$$\int \ln |r - |\zeta||^{-1} d\mu(\zeta) \geqslant \int \ln |z - \zeta|^{-1} d\mu(\zeta)$$
$$\geqslant (1 + A\epsilon) P_\mu(0).$$

Hence

$$\begin{aligned}(1 + A\epsilon) P_\mu(0) &\leqslant \int_\beta^\alpha \left[\int \ln |r - |\zeta||^{-1} d\mu(\zeta)\right] \epsilon(r \ln r^{-1})^{-1} dr \\ &= \int \left[\int_\beta^\alpha \ln |r - |\zeta||^{-1} \epsilon(r \ln r^{-1})^{-1} dr\right] d\mu(\zeta) \\ &< (1 + A\epsilon) \int \ln |\zeta|^{-1} d\mu(\zeta) \text{ by 5.2} \\ &= (1 + A\epsilon) P_\mu(0).\end{aligned}$$

This contradiction proves the lemma.

We require some simple results from general topology. (For metric spaces 5.4 is proved in [14], iv 5 Ex 1 and 5.5 (1) in [13] 42, iii 3).

5.4. LEMMA. *Let F be a compact Hausdorff space and Z a closed subset of F. Then the union Y of all components of F meeting Z is closed.*

Proof. Assume $F \neq Y$. Then fix $y \in F - Y$; for each $x \in Z$, the component of F containing x does not contain y, so $\exists$ an open and closed subset G_x of F containing x but not y. Then we can find a finite set $x_1 \cdots x_n \in Z$ such that $Z \subseteq \bigcup G_{x_i} = G$ say. Since G is open and closed it contains Z, so F–G is a neighbourhood of y not meeting Z. Thus Z is closed.

5.5. LEMMA. *Let F be a compact connected Hausdorff space.*

(1) *If U is a non-empty open proper subset of F, every component of $F - U$ meets $\bar{U}$.*

(2) *If U_1 and U_2 are disjoint non-empty open subsets of F, some component of $F - (U_1 \cup U_2)$ meets both $\bar{U}_1$ and $\bar{U}_2$.*

Proof. (1) Let Z be a component of $F - U$ and suppose $Z \cap \bar{U} = \phi$. Then there is an open and closed subset G of $F - U$ not meeting $\partial_F U$ but containing Z. Then G is a non-empty open and closed proper subset of F, contradicting the connectedness of F.

(2) By (1) each component of $F - (U_1 \cup U_2)$ meets $\overline{U_1 \cup U_2} = \bar{U}_1 \cup \bar{U}_2$ so meets either $\bar{U}_1$ or $\bar{U}_2$. Suppose no component meets both. For $i = 1, 2$ let Z_i be the union of all components of $F - (U_1 \cup U_2)$ which meet $\partial_F U_i$. Z_1 and Z_2 are disjoint and closed in $F - (U_1 \cup U_2)$ by 5.4. Then the sets $P_i = U_i \cup Z_i = \bar{U}_i \cup Z_i$ are disjoint and closed in F, and $F = P_1 \cup P_2$. Also, since F is connected, $\partial_F U_1$ and $\partial_F U_2$ are non-empty, so P_1 and P_2 are non-empty. This contradicts the connectedness of F.

For $0 < t < e^{-1}$ put $\varphi(t) = (\ln \ln t^{-1})^{-1}$ and let m_φ denote the associated Hausdorff measure.

5.6. LEMMA. *Suppose $\bar{\Omega}$ is connected and $I \subseteq F \cup G$ where F is a compact subset of X with $m_\varphi(F) = 0$ and G is countable. Let $y \in K^0$ and let μ be a representing measure for y. Then $\mu = \lambda_y$.*

Proof. If F is empty the result follows from 5.1, so assume $F \neq \phi$. Put $g(z) = P_\mu(z) - \ln | z - y |^{-1}$. By 5.1, g is constant on $\bar{\Omega}_i$ for each i; we wish to show that all these constants are equal (and hence zero). So fix p, q and let $g = \gamma$ on Ω_p, $g = \delta$ on Ω_q. We shall prove $\gamma = \delta$.

Choose ϵ, $0 < \epsilon < 1$, and ρ such that

$$0 < \rho < \min(\epsilon, \tfrac{1}{4}, \tfrac{1}{2}\,\text{diameter}(\Omega_q), \tfrac{1}{4}\, d(X, y)). \tag{1}$$

Choose open discs $\Delta_1 \cdots \Delta_m$ with radii $r_1 \cdots r_m$ so that $F \subseteq \bigcup_{i=1}^m \Delta_i$, $\Delta_i \cap F \neq \phi$ for each i, and

$$\sum_{i=1}^{m} (\ln \ln r_i^{-1})^{-1} < \tfrac{1}{2}\,\rho.$$

Since P_μ is lower semi-continuous, $\exists x_i \in \bar{\Delta}_i$ so that $g(x_i) \leqslant g(x)$, $\forall x \in \bar{\Delta}_i$. Put $\delta_i = 2(\ln \ln r_i^{-1})^{-1}$, $\beta_i = 2r_i$, $\ln \ln \alpha_i^{-1} = \ln \ln \beta_i^{-1} - \delta_i^{-1}$. Then we have

$$\ln \alpha_i^{-1} = \exp \delta_i^{-1} - e^{-\delta_i^{-1}} \ln 2 > \delta_i^{-1}$$

since $\delta_i < \frac{1}{2}$. So $\alpha_i < \delta_i < \frac{1}{4}$.

Applying 5.5 we obtain t_i, $\beta_i < t_i < \alpha_i$, such that

$$P_\mu(z) < P_\mu(x_i)\,(1 + A\delta_i) + m\delta_i \qquad \text{for} \qquad |\, z - x_i \,| = t_i\,, \tag{2}$$

where m is a constant depending only on K. We have

$$t_i < \delta_i \qquad \text{and} \qquad \sum_{i=1}^{m} t_i < \sum_{i=1}^{m} \delta_i < \rho. \tag{3}$$

Denote by D_i the disc$\{z : |\, z - x_i \,| = t_i\}$ and by Γ_i its boundary. We have, using (3) and (1)

$$|\, x_i - y \,| \geqslant d(y, K) - d(x_i\,, K) \geqslant d(y, K) - 2\rho \geqslant \tfrac{1}{2}\, d(y, K). \tag{4}$$

Also, if $z \in \Gamma_i$ then

$$\begin{aligned} |\ln |\, z - y \,| - \ln |\, x_i - y \,|\,| &= |\ln |\, 1 - (x_i - y)^{-1}\,(x_i - z)|\,| \\ &\leqslant 2t_i\, |\, x_i - y \,|^{-1} \leqslant 4t_i\, d(y, K)^{-1} \end{aligned}$$

by (4). Hence if $z \in \Gamma_i$, using (2),

$$g(z) \leqslant g(x_i) + (AP_\mu(x_i) + m)\,\delta_i + 4t_i\, d(y, K)^{-1}. \tag{5}$$

Put

$$M = A(|\,\gamma\,| + 1 + \ln^+(2d(y, K)^{-1})) + m + 4d(y, K)^{-1} \tag{6}$$

and assume henceforth that $\epsilon < M^{-1}$.

We next define inductively a sequence $i_1 \cdots i_k$ of distinct integers from the set $\{1, 2 \cdots m\}$ with the property that for $z \in \Gamma_{i_s}$,

$$g(z) \leqslant \gamma + M \sum_{t=1}^{s} \delta_{i_t} \leqslant \gamma + 1, \qquad 1 \leqslant s \leqslant k. \tag{7}$$

where the second inequality follows from $\sum_{i=1}^{m} \delta_i < \rho \leqslant \epsilon < M^{-1}$. If $\bar{\Omega}_p$ meets $\bar{\Delta}_i$ for some i, we take i_1, to be any such i. If not, let Y be the component of $\bar{\Omega} - \bigcup_{i=1}^{m} \Delta_i$ containing $\bar{\Omega}_p$; then by 5.5 (1) we can choose i_1 so that Y meets $\bar{\Delta}_{i_1}$. Since $F \cap Y = \phi$, by 5.1 g is constant on Y. Thus in either case $g(x_{i_1}) \leqslant Y$, so that

$$P(x_{i_1}) \leqslant \gamma + \ln | x_{i_1} - y |^{-1} \leqslant \gamma + \ln^+(2d(y, K))^{-1}$$

using (4). So by (5) and (6), for $z \in \Gamma_{i_1}$, $g(z) \leqslant \gamma + M\delta_{i_1}$, i.e. (7) is valid for $s = 1$.

Now suppose $i_1 \cdots i_s$ chosen, satisfying (7). Write $D = \bigcup_{t=1}^{s} D_{i_t}$. If $\bigcup_{i=1}^{m} \Delta_i \subseteq D$, then the construction terminates and $s = k$. If not, let $J = \{i; 1 \leqslant i \leqslant m, \Delta_i \not\subseteq D\}$, so that $J \neq \phi$. We consider two cases: (a) $\exists i \in J$ with $\bar{\Delta}_i \cap \bar{D} \neq \phi$ and (b) $\exists$ no such i. In case (a), we take i_{s+1} to be any $i \in J$ with $\bar{\Delta}_i \cap \bar{D} \neq \phi$. Since $\Delta_i \not\subseteq D$, $\bar{\Delta}_i$ must meet $\bigcup_{t=1}^{s} \Gamma_{i_t}$, so that by (7)

$$g(x_{i_{s+1}}) \leqslant \gamma + M \sum_{t=1}^{s} \delta_{i_t} \leqslant \gamma + 1. \tag{8}$$

In case (b) we apply 5.5 (2) to the compact connected space $\bar{\Omega}$ and the non-empty relatively open subsets $D \cap \bar{\Omega}$ and $(\bigcup_{i \in J} \Delta_i) \cap Q$, and obtain a compact connected subset Y of $\bar{\Omega}$, not meeting $\bigcup_{i=1}^{m} \Delta_i$ but meeting Γ_{i_r} for some r, $1 \leqslant r \leqslant t$, and meeting $\bar{\Delta}_i$ for some $i \in J$. We take i_{s+1} to be this i. By 5.1 g is constant on Y, so that (8) holds in this case also. Thus in either case by (4) and (8)

$$P_\mu(x_{i_{s+1}}) \leqslant \gamma + 1 + \ln^+(2d(y, K)^{-1}),$$

and by (5) and (6), for $z \in \Gamma_{i_{s+1}}$,

$$g(z) \leqslant \gamma + M \sum_{t=1}^{s} \delta_{i_t} + M\delta_{i_{s+1}} = \gamma + \sum_{t=1}^{s+1} M\delta_{i_t}.$$

So (7) holds for $s + 1$. This completes the construction which clearly terminates with $k \leqslant m$. Then

$$\bigcup_{t=1}^{k} D_{i_t} \supseteq \bigcup_{i=1}^{m} \Delta_i \supseteq F, \qquad \text{and for} \qquad z \in \bigcup_{t=1}^{k} \Gamma_{i_t},$$
$$g(z) \leqslant \gamma + M \sum_{t=1}^{k} \delta_{i_t} \leqslant \gamma + \epsilon M. \tag{9}$$

By (1) the diameter of each D_i is less than that of Ω_q, so if $\bigcup_{t=1}^{k} D_{i_t}$ meets $\bar{\Omega}_q$, then $\bigcup_{t=1}^{k} \Gamma_{i_t}$ meets $\bar{\Omega}_q$ so that by (9) $\delta \leqslant \gamma + \epsilon M$. If $\bigcup_{t=1}^{k} D_{i_t}$ does not meet $\bar{\Omega}_q$, the component Z of $Q - \bigcup_{t=1}^{k} D_{i_t}$ containing $\bar{\Omega}_q$ meets $\bigcup_{t=1}^{k} \Gamma_{i_t}$ by 5.5 (1). g is constant on Z by 5.1, so by (9) again $\delta \leqslant \gamma + \epsilon M$. So in either case $\delta \leqslant \gamma + \epsilon M$. Let $\epsilon \to 0$; thus $\delta \leqslant \gamma$ and similarly $\gamma \leqslant \delta$, so $\gamma = \delta$ as required.

Hence $g = 0$ on $\bigcup_i \bar{\Omega}_i$. Since $F \cup G$ has 1-dimensional measure zero, it follows from 1.5 that $g = 0$ on $\bar{\Omega}$ i.e. $P_\mu(z) = \ln | z - y |^{-1}$ for $z \in \bar{\Omega}$. In particular this applies to λ_y, so $P_{\lambda_y}(z) = \ln | z - y |^{-1}$, $z \in \bar{\Omega}$, for all $y \in K^0$. Thus if $y_0 \in K^0$,

$$\begin{aligned} P_\mu(y_0) &= \int \ln | z - y_0 |^{-1} \, d\mu(z) \\ &= \iint \ln | z - \zeta |^{-1} \, d\mu(z) \, d\lambda_{y_0}(\zeta) \\ &= \int \ln | \zeta - y |^{-1} \, d\lambda_{y_0}(\zeta) \\ &= P_{\lambda_y}(y_0) \end{aligned}$$

since we can replace μ by λ_y. Thus $P_\mu = P_{\lambda_y}$ throughout $\mathbf{C}$, so $\mu = \lambda_y$ by [8] Lemma 2.

Note that the last part of the proof could be shortened by appealing to Theorem 2.2; however we preferred to keep this section independent of that theory.

5.7. THEOREM. *Under the hypotheses of Lemma* 5.6, *R*(*K*) *is a Dirichlet algebra on X.*

Proof. We use the following results from abstract H^p-theory. Let A be a uniform algebra on a compact space X, and let φ be a multiplicative functional on A with unique representing measure λ on X. Then if $\mu \in A^\perp$, we have the Lebesgue decomposition $\mu = \rho + \sigma$, $\rho \ll \lambda$, $\sigma \perp \lambda$. By [21], ρ and $\sigma \in A^\perp$. Then $\rho = f\lambda$ for some $f \in H_0^1(\lambda)$, the closure in $L^1(\lambda)$ of $\{g \in A : \int g \, d\lambda = 0\}$ [16]. If μ is real then ρ and hence f are real, so $f = 0$ by [16]. Thus $\mu = \sigma$ is singular w.r.t. λ if $\mu \in \operatorname{Re} A^\perp$.

Applying this to $R(K)$ on X, if $\mu \in \operatorname{Re} R(K)^\perp$ and $y \in K^0$ then $\mu \perp \lambda_y$. Just as in [1], Section 5, Lemma, using [21] it follows that

$$\int \frac{d\mu(\zeta)}{z - \zeta} = 0, \qquad z \in \mathbf{C} - X.$$

As in [8], Lemma 4

$$\int \frac{d\mu(\zeta)}{z - \zeta} = 0 \qquad \text{for} \qquad z \in \partial\Omega_i$$

provided

$$\int \frac{d\,|\,\mu\,|(\zeta)}{|\,z-\zeta\,|} < \infty, \qquad i = 1, 2, \ldots .$$

Thus $\int d\mu(\zeta)/(z-\zeta) = 0$ a.e. (plane measure) so by [8], Lemma 5, $\mu = 0$.

6. *Annihilating measures-general case.*

In this section we combine the results of Sections 2 and 5 to obtain information about real annihilating measures, along the lines of [3], the methods of which we follow closely.

6.1. LEMMA. *Suppose Q is a component of $\bar{\Omega}$ with more than one point, such that $Q \cap I$ is totally disconnected. Then $Q = (Q \cap \Omega)^-$ and $\partial(Q \cap \Omega) = Q \cap X$.*

Proof. Since I is totally disconnected and Q connected with more than one point, $Q - I$ is dense in Q, i.e. $Q = \{Q \cap (\bigcup_i \bar{\Omega}_i)\}^-$. For any i, if $\bar{\Omega}_i$ meets Q then $\bar{\Omega}_i \subseteq Q$; thus $Q \cap (\bigcup_i \Omega_i)$ is dense in Q, so $Q = (Q \cap \Omega)^-$.

$$\begin{aligned}\partial(Q \cap \Omega) &= (Q \cap \Omega)^- - (Q \cap \Omega) = Q - (Q \cap \Omega)\\ &= Q \cap (\bar{\Omega} - \Omega) = Q \cap X.\end{aligned}$$

6.2. THEOREM. *Let Q be a component of $\bar{\Omega}$ with more than one point and $Q \cap I \subseteq F \cup G$, where G is countable and F compact with $m_\varphi(F) = 0$ (φ as in Section 5). Let $\mu \in \operatorname{Re} R(K)^\perp$. Then P_μ is constant on $Q \cap \Omega$.*

Proof. Put $L = S - (Q \cap \Omega)$ ($S =$ complex sphere). Then by 6.1, $Q = (S - L)^-$ and $\partial L = Q \cap X$. Let ν be the measure obtained by "sweeping" μ to ∂L, i.e. for $f \in C(\partial L)$, $\int f\, d\nu = \int \tilde{f}\, d\mu$ where $\tilde{f}$ is defined on L by $\tilde{f} = f$ on ∂L and on L^0, $\tilde{f}$ is the harmonic extension of f to L^0 (where L^0 is the interior relative to S).

If $f \in R(L)$, $\tilde{f} = f$ and so $\int f\, d\nu = \int f\, d\mu = 0$, since $f \mid K \in R(K)$. Thus $\nu \in \operatorname{Re} R(L)^\perp$ and by 5.7 $\nu = 0$. Let $z_1, z_2 \in Q \cap \Omega$, and put $g(\zeta) = \ln |\,\zeta - z_1\,|^{-1} - \ln |\,\zeta - z_2\,|^{-1}$, so g is harmonic in a neighborhood of L. Then $\tilde{g} = g$, so $\int g\, d\mu = \int g\, d\nu = 0$, i.e. $P_\mu(z_1) = P_\mu(z_2)$ as required.

Let $Q_0, Q_1, \ldots$ be the components of $\bar{\Omega}$ which meet Ω, Q_0 being the unbounded one. We suppose that for $j > 0$, $Q_j \cap (\bar{\Omega} - Q_j)^- = \phi$. Let $\eta_j = \lambda(Q_j)$ as constructed in Section 4 ($j > 0$). Then $P_{\eta_j} = 1$ on $Q_j \cap \Omega$, $P_{\eta_j} = 0$ on $\Omega - Q_j$. Following [3] we denote by s^n the operation of sweeping a measure to $\bigcup_{k=0}^{n} \partial Q_k$ and write $\eta_j{}^n = s^n \eta_j$,

$1 \leqslant j \leqslant n$. If $0 \leqslant k \leqslant n$, $1 \leqslant j \leqslant n$, then for $z \in \Omega \cap Q_k$ we have $P_{\eta_j}(z) = P_{\eta_j{}^n}(z) = \delta_{kj}$. Let Σ denote the weak*-closed real linear span of the measures η_j.

6.3. THEOREM. *Suppose* $\bar{H}(X) = C_R(X)$, $\lambda_y(X - P) = 0$ *for* $\forall_y \in K^0$, *and (with notation as above)* $Q_j \cap (\bar{\Omega} - Q_j)^- = \phi$, $j > 0$. *Suppose also that for* $i \geqslant 0$, $Q_i \cap I \subseteq F_i \cup G$, *where* G *is countable and for each* i, F_i *is compact with* $m_\varphi(F_i) = 0$ (φ *as in Section* 5). *Suppose further that for real sequences* $\{c_j\}$,

$$\sup_n \left\| \sum_{i=1}^n c_j \eta_j{}^n \right\| < \infty \Rightarrow \liminf_n \left\| \sum_{i=1}^n c_j \eta_j \right\| < \infty.$$

Then

$$\operatorname{Re} R(K)^\perp = \Sigma + D(K)^\perp \qquad \textit{and} \qquad R(K) = A(K).$$

Proof. Trivially

$$\Sigma + D(K)^\perp \subseteq \operatorname{Re} R(K)^\perp.$$

To prove the converse, let

$$K_n = \mathbf{C} - \bigcup_{k=0}^n (Q_k \cap \Omega), \qquad n \geqslant 1.$$

Then by 6.1,

$$(\mathbf{C} - K_n)^- = \bigcup_{k=0}^n Q_k \qquad \text{and} \qquad \partial K_n = \bigcup_{k=0}^n \partial Q_k\,.$$

So s^n is the operation of sweeping a measure to ∂K_n. Let $\mu \in \operatorname{Re} R(K)^\perp$. Then by 6.2, P_μ is constant, say c_j, on $Q_j \cap \Omega$ for $j \geqslant 1$ and $P_\mu = 0$ on $Q_0 \cap \Omega$. Then $P_{s^n\mu} = c_j$ on $Q_j \cap \Omega$ if $j \leqslant n$. Let

$$\nu = s^n\mu - \sum_{j=1}^n c_j \eta_j{}^n.$$

Then $P_\nu = 0$ in $\bigcup_{k=0}^n (Q_k \cap \Omega)$, i.e. outside K_n. It is easily seen that K_n satisfies the hypotheses of Theorem 2.2, and all points of ∂K_n are regular, so that $\nu = 0$. Hence $s^n\mu = \sum_{j=1}^n c_j \eta_j{}^n$ and so $\| \sum_{j=1}^n c_j \eta_j{}^n \| \leqslant \| \mu \|$. By hypothesis $\liminf_n \| \sum_{j=1}^n c_j \eta_j \| < \infty$ so $\{\sum_{j=1}^n c_j \eta_j\}$ has a subsequence converging weak* to a measure η on X. Then $\eta \in \Sigma$ and for $z \in \Omega \cap Q_k$, $P_\eta(z) = c_k$. Thus $P_{\mu-\eta} = 0$ in Ω, so that by Theorem 2.2, $\mu - \eta \in D(K)^{\perp}$ and $\mu \in D(K)^{\perp} + \Sigma$ as required.

Since $D(K)^{\perp} + \Sigma \subseteq \operatorname{Re} A(K)^{\perp}$, it follows that $\operatorname{Re} R(K)^{\perp} = \operatorname{Re} A(K)^{\perp}$ and so from [2] $R(K) = A(K)$.

6.4. COROLLARY. *If $\bar{\Omega}$ has exactly $n + 1$ components and $I \subseteq F \cup G$ where G is countable and F compact with $m_{\varphi}(F) = 0$, then* $\operatorname{Re} R(K)^{\perp}$ *has dimension n and $R(K) = A(K)$.*

Proof. It is easily verified that K satisfies the hypotheses of Theorem 6.3, and that $D(K) \mid X = C_R(X)$ (i.e. K^0 is regular). Then $\operatorname{Re} R(K)^{\perp} = \Sigma$, which is spanned by $\eta_1 \cdots \eta_n$.

Notes. (1) Conditions under which the last hypothesis of Theorem 6.3 is fulfilled can be obtained by the methods of [3], Section 2. For example, one obtains the following:

Suppose that K^0 is connected and dense in K, that $\lambda_y(X - P) = 0$ for $y \in K^0$, and $Q_i \cap (\bar{\Omega} - Q_j)^- = \phi$, for $j > 0$. Suppose also $Q_i \cap I \subseteq F_i \cup G$ where G is countable and F_i is compact with $m_{\varphi}(F_i) = 0$. Suppose further that there exists $\delta > 0$ such that for each $i \geqslant 1$, ∂Q_i can be separated from the rest of X by a Jordan curve γ_i in K satisfying

$$\lambda_z(\partial Q_0) \geqslant \delta, \qquad z \in \gamma_i .$$

Then

$$\operatorname{Re} R(K)^{\perp} = \Sigma + D(K)^{\perp} \qquad \text{and} \qquad R(K) = A(K).$$

The last hypothesis will be satisfied if the diameters of the Q_i tend to zero sufficiently fast, relative to the distances between them.

(2) Corollary 6.4 can be proved directly by the methods of Section 5, but requires the more involved H^p theory of [9] to replace the proof of 5.7. The theory of [9] together with a decomposition of the type of [1], Theorem 1, provides a description of all (not necessarily real) annihilating measures, as follows: $\mu \in R(K)^{\perp} \Leftrightarrow \mu$ has the form:

$$\mu = \sum_{i=1}^{\infty} f_i \lambda_{y_i} + \sum_{j=1}^{n} c_j \eta_j$$

where $y_i \in K^0$, $f_i \in H_0^1(\lambda_{y_i})$ (see 5.7), $\sum_i \| f_i \|_{L^1(\lambda_{y_i})} < \infty$ and $c_j \in \mathbf{C}$.

Corollary 6.4 can also be deduced from Theorem 5.7 and [4], Lemma 3.

(3) Other results in this direction can be found in [4].

7. *Examples.*

7.1. We first consider an example where ∂K (and hence $\bar{\Omega}$) is connected. The example was considered in [7] and is obtained by taking a perfect nowhere dense subset E of the closed interval $[-1, 1]$ and forming K by removing from the closed unit disc those open discs whose diameters are complementary intervals of E.

We suppose E contains 1 and -1. Since I lies on the real axis, by [17] Theorem 5 we have $R(K) = A(K)$.

Let U_1, U_2 denote the components of K^0. Then harmonic measure for $y \in U_i$ is mutually absolutely continuous w.r.t. arc length on ∂U_i since ∂U_i is rectifiable. We assert that $R(K)$ is Dirichlet if and only if E has zero length. To prove this we note that if $f \in C(K)$ then $f \in A(K)$ if and only if $f \mid \bar{U}_i \in A(\bar{U}_i)$, $i = 1, 2$. Further $A(U_i)$ is isomorphic with the disc algebra by a conformal map so that by the F. and M. Riesz theorem E is a null set for all annihilating measures, if E has zero length. Now let $u \in C_R(\partial K)$. By the Rudin-Carleson theorem (e.g. [22], Corollary 3.2) applied to U_1, U_2, $\exists g \in A(K)$ with $g = u$ on E. If μ is a real measure on ∂U_i orthogonal to $\{f \in A(U_i) : f = 0 \text{ on } E\}$ then by [22] Lemma 4.1, $\mu = \rho + \sigma$ where $\rho \in A(\bar{U}_i)^\perp$ and σ is a measure on E. Then $\rho \mid E = 0$ so ρ is real, hence $\rho = 0$. Thus $\int (g - u)\, d\mu = 0$ for all such μ, so for any $\epsilon > 0$, $\exists f_i \in A(\bar{U}_i)$ with $f_i = 0$ on E and $|\operatorname{Re}(f_i + g - u)| < \epsilon$ on ∂U_i. Define $f = f_i$ on $\bar{U}_i$; then $f + g \in A(K)$ and $|\operatorname{Re}(f + g) - u| < \epsilon$ on K. So $R(K)$ is dirichlet.

Conversely if E has positive length then U_1 and U_2 are in the same Gleason part for $R(K)$ since harmonic measures for points in U_1, U_2 are never mutually singular ([20]).

But for a Dirichlet algebra, every part with more than one point is of the form $\varphi(\Delta)$ where Δ is the unit disc, φ is $(1-1)$ and $f_0\varphi$ is analytic for $f \in A$, in particular for $f(z) \equiv z$ [19]. Then $U_1 \cup U_2$ must be contained in a connected open subset of K, which is impossible. So $R(K)$ cannot be dirichlet.

This suggests the following conjecture: if $\bar{\Omega}$ is connected and $\lambda_y(I) = 0$, $\forall y \in K^0$, then $R(K)$ is dirichlet.

We now show that E can be chosen so that every point of ∂K is a peak point for $R(K)$ (and hence in P), but that E has positive linear measure so that $R(K)$ is not dirichlet. It suffices to construct E so that

$$\sum_{n=1}^{\infty} 2^n m[(x - 2^{-n}, x + 2^{-n}) - E] = \infty, \qquad \text{for all} \qquad x \in E. \tag{1}$$

In fact, using the relation

$$\sum_{n=1}^{\infty} 2^n m((x - 2^{-n}, x + 2^{-n}) - E) = \sum_{n=1}^{\infty} (2^{n+1} - 2)\, m(A_n(x) \cap R - E)$$

where R is the real axis and $A_n(x)$ is the annulus

$$A_n(x) = \{z : 2^{-(n+1)} \leqslant |z - x| \leqslant 2^{-n}\},$$

(1) yields

$$\sum_{n=1}^{\infty} 2^n m(R \cap A_n(x) - E) = \infty.$$

For linear sets S, $\gamma(S) \geqslant \frac{1}{4} m(S)$ where γ denotes analytic capacity (see [23], Proposition 3.9). Thus

$$\sum_{n=1}^{\infty} 2^n \gamma(R \cap A_n(x) - E) = \infty.$$

Hence

$$\sum_{n=1}^{\infty} 2^n \gamma(A_n(x) - K) = \infty$$

since γ is monotone and $R - E$ does not meet K. By Melnikov's criterion ([23], Theorem 6.1) each point of K is a peak point for $R(K)$.

We now construct E satisfying (1). We define by induction on i a system of open intervals I_{ij} ($i = 1, 2,...; j = 1, 2,..., n_i$) with disjoint closures contained in $(-1, 1)$, so that

$$\sum_{j=1}^{n_i} m(I_{ij}) < 2^{-i}$$

and

$$\sum_{j=1}^{n_i} \frac{m(I_{ij})}{\rho_{ij}(x)} \geqslant 1 \qquad \text{for all} \qquad x \in [-1, 1] - \bigcup_{k=1}^{i} \left(\bigcup_{j=1}^{n_k} I_{kj} \right) \tag{2}$$

where $\rho_{ij}(x)$ is the distance from x to the farthest point (from x) of $\bar{I}_{ij}$. We require a simple lemma:

7.1.1. LEMMA. *Let I be an open interval and let $\epsilon > 0$. Then we* *can find intervals* $I_1, \ldots, I_n$ *with disjoint closures contained in* I, *so that*

$$\sum_{i=1}^{n} m(I_i) < \epsilon$$

and

$$\sum_{i=1}^{n} \frac{m(I_i)}{\rho_i(x)} \geqslant 1, \qquad x \in \bar{I} - \bigcup_{i=1}^{n} I_i$$

where $\rho_i(x)$ *is the distance from* x *to the farthest point of* $\bar{I}$.

Proof. We can suppose I is the unit interval. With n an integer to be chosen, let I_i $(1 \leqslant i \leqslant n)$ be the interval with centre $i/n+1$ and length ϵ/n (we may suppose $\epsilon < \frac{1}{2}$, so that the $\bar{I}_i$ are disjoint). For $x \in \bar{I} - \bigcup_{i=1}^{n} I_i$, we have

$$\sum_{i=1}^{n} \frac{m(I_i)}{\rho_i(x)} \geqslant \frac{\epsilon}{n}\left(\frac{n+1}{2} + \frac{n+1}{3} + \cdots + 1\right)$$

$$\geqslant \epsilon \frac{n+1}{n}(\ln(n+1) - 1) \geqslant 1$$

if n is large enough.

It is now easy to construct I_{ij}; one applies the lemma to each component interval of $(-1, 1) - \bigcup_{k=1}^{i-1} (\bigcup_{j=1}^{n_k} \bar{I}_{kj})$ with $\epsilon < N^{-1}\, 2^{-i}$, where N is the number of these intervals. Then $\{I_{ij}\}_{j=1}^{n_i}$ is the totality of new intervals obtained in this way. (2) follows immediately from the lemma. It is clear that we can suppose $\bigcup_{i,j} I_{ij}$ dense in $(-1, 1)$.

Put $E = [-1, 1] - \bigcup_{i,j} I_{ij}$. Then (1) is a consequence of (2) and $m(E) = 2 - \sum_{i,j} m(I_{ij}) \geqslant 1$. This completes the construction.

Since peak points are one-point parts, in this example $U_1 \cup U_2$ is a disconnected Gleason part for $R(K)$. (This example has been discovered independently by J. Garnett.)

The above construction was suggested by an example of J. Wermer [24]. *Note added 18th August 1969.* T. W. Gamelin and J. Garnett have found a construction of the above example which avoids the use of Melnikov's theorem and the concept of analytic capacity; see Lemma 5.3 of "Distinguished Homomorphisms and Fiber Algebras" by the above authors (preprint).

Finally we note that, as shown in [7], E can be chosen so that $X - P$ has positive harmonic measure and so the set of Jensen measures for $A(K)$ need not be norm compact. In this case roughly speaking "analytic" approximation is possible $(R(K) = A(K))$ but "harmonic" approximation is not $(\bar{H}(K) \neq D(K))$.

7.2. We now construct an example exhibiting the last phenomenon, in which each component of $\bar{\Omega}$ is "nice". Let Δ denote the open unit disc and let E be any compact subset of Δ with positive logarithmic capacity but zero analytic C-Capacity [23] and $\mathbf{C} - E$ connected. Let $x_1, x_2 \cdots$ be a sequence of points of $\Delta - E$ whose set of limit points is exactly E. We choose $r_1, r_2, \ldots$ so that the closures of the discs $\Delta_i = \{z : | z - x_i | < r_i\}$ are disjoint subsets of $\Delta - E$. We define $K = \bar{\Delta} - \bigcup_i \Delta_i$ so that $X = \partial K = \partial \Delta \cup (\bigcup_i \partial \Delta_i) \cup E$. From [7] 6.6(iv) it follows easily that $E \cap P = \phi$ if $r_i \to 0$ fast enough. Then $R \neq P$ so $\bar{H}(K) \neq D(K)$. But K^0 is connected, so $\bar{H}(X) = C_R(X)$ and so $X - P$ has positive harmonic measure. On the other hand, [18], Theorem 4 shows that $R(K) = A(K)$ if $r_i \to 0$ fast enough.

7.3. We use a modification of the above construction to show that the last hypothesis in Theorem 6.3 cannot be dropped. Again let Δ be the open unit disc, and now let E be a compact subset of Δ with positive analytic C-Capacity, $E^0 = \phi$, and $\mathbf{C} - E$ connected. We suppose also $E \subseteq \{z : | z | \leqslant \frac{1}{2}\}$. For each positive integer r we can find Jordan domains with smooth boundaries $U_{r,1} \cdots U_{r,n_r}$ such that $n_r \leqslant 2^{2r}$; diameter $(U_{r,i}) = 2^{-5r}$; $d(U_{r,i}, U_{r,j}) \geqslant 2^{-r-1}$ for $i \neq j$; for $z \in E$, $\exists i$, $1 \leqslant i \leqslant n_r$, with $d(z, U_{r,i}) \leqslant 2^{-r}$; $d(U_{r,i}, E) \leqslant 2^{-r}$; $\bar{U}_{r,i} \subseteq \Delta - E$. We then choose a sequence $r_1 < r_2 < \cdots$ so that

$$2^{-r_{k+1}} < \tfrac{1}{2} d\left(E, \bigcup_{i=1}^{n_{r_k}} \bar{U}_{r,i}\right).$$

Now put

$$K = \Delta - \bigcup_{k=1}^{\infty} \left(\bigcup_{i=1}^{n_{r_k}} U_{r_k,i}\right).$$

Then

$$\partial K = \partial \Delta \cup \left(\bigcup_{k=1}^{\infty} \bigcup_{i=1}^{n_{r_k}} \partial U_{r_k,i}\right) \cup E$$

and it follows easily from [7] 6.6.(iv) that $E \subseteq P$, hence $X = P$ since the U_i have "nice" boundaries. On the other hand Theorem 4′ of [18] ensures that $A(K) \neq R(K)$. The $Q_1, Q_2, \ldots$ of Theorem 6.3 are just $\bar{U}_1, \bar{U}_2, \ldots$ and satisfy the conditions in Theorem 6.3, except the last, which therefore cannot be dropped.

Note added in proof. Since this paper was written the author has strengthened Theorem 5.7; the only restriction needed on I is that it have zero $\frac{1}{2}$-dimensional Hausdorff measure.

The conjecture in Section 7.1 is false as it stands. See the final two paragraphs of *Dirichlet algebras of analytic functions* by the present author, to appear in this Journal.

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References

1. I. Glicksberg and J. Wermer, Measures orthogonal to a dirichlet algebra, *Duke Math. J.* **30** (1963), 661–666.
2. I. Glicksberg, The abstract F. and M. Riesz theorem, *J. Functional Analysis* **1** (1967), 109–122.
3. J. Garnett and I. Glicksberg, Algebras with the same multiplicative measures, *J. Functional Analysis* **1** (3) (1967), 331–341.
4. I. Glicksberg, Dominant representing measures and rational approximation, *Trans. Amer. Math. Soc.* **130** (1968), 425–462.
5. P. R. A. Ahern and D. Sarason, On some hypodirichlet algebras of analytic functions, *Amer. J. Math.* (1968), 932–941.
6. T. A. McCullough, Rational approximation on certain plane sets, *Pacific J. Math.* **29** (1969), 631-639.
7. T. W. Gamelin and H. Rossi, "Jensen Measures" in Function Algebras, Proc. of conference, Tulane 1965. Scott Foresman, 1966, pp. 13–35.
8. L. Carleson, Mergelyan's theorem on uniform approximation, *Math. Scand.* **15** (1964), 167–175.
9. P. R. Ahern and D. Sarason, The H^p spaces of a class of function algebras, *Acta Math.* **117** (1967), 123–163.
10. M. Brelot, "Theorie Classique du Potentiel," Les Cours de Sorbonne, Paris, 1961.
11. R. Phelps, "Lectures on Choquet's Theorem," Van Nostrand.
12. L. V. Ahlfors and L. Sario, "Riemann Surfaces," Princeton, N. J., 1960.
13. C. Kuratowski, "Topologie," Polskie Touarzystwo Matematyczne, Warsaw, 1952, Vol. 2.
14. M. H. A. Neumann, "Topology of Plane Sets," Cambridge Univ. Press, 1954 (2nd. Ed.).
15. D. R. Wilken, Lebesgue measure for parts of $R(X)$, *Proc. Amer. Math. Soc.* **18** (1967).
16. G. Lumer, Herglotz transformation and H^p theory, *Bull. Amer. Math. Soc.* **71** (1965), 725–730.
17. M. S. Melnikov, Analytic capacity and the cauchy integral, *Soviet Math.* **8** (1967), 20–23.

18. E. P. Dolzhenko, Approximation on closed regions and zero-sets, *Soviet Math.* **3** (1962), 472–475.
19. J. Wermer, Dirichlet algebras, *Duke Math. J.* **27**, 372–382.
20. E. Bishop, Representing measures for points in a uniform algebra, *Bull. Amer. Math. Soc.* **70** (1966), 121–122.
21. P. R. Ahern, On the generalized F. and M. Riesz theorem, *Pacific J. Math.* **15** (1965), 373–376.
22. I. Glicksberg, Measures orthogonal to algebras and sets of antisymmetry, *Trans. Amer. Math. Soc.* **105** (1962), 415–435.
23. L. Zalcman, "Analytic Capacity and Rational Approximation," Lecture notes in Math. No. 50, Springer-Verlag, Berlin, 1968.
24. J. Wermer, Bounded point derivations on certain banach algebras, *J. Functional Analysis* **1** (1967), 28–36.